Recall

<u>Weyl's Criterion</u>

A sequence $(x_n)$ in $[0,1)$ is equidistributed if and only if for any $k \in \mathbb{Z} \setminus \{0\}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k x_n} \to 0 \quad \text{as } N \to \infty$$

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- $\langle n\alpha \rangle$ is equidistributed if $\alpha \notin \mathbb{Q}$
- $\langle n^2\alpha \rangle$ is equidistributed if $\alpha \notin \mathbb{Q}$

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The idea of the proof is to reduce the degree since $(x+h)^2 - x^2 = 2hx + h^2$

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I. Let $f: \mathbb{N} \to \mathbb{R}$ be a function and $S_N = \sum_{n=1}^{N} e^{2\pi i f(n)}$.

Then $|S_N|^2 \leq 4 \frac{N}{H} \sum_{h=0}^{H-1} \left| \sum_{n=1}^{N-h} e^{2\pi i (f(n+h) - f(n))} \right|$ for $H \leq N$.

Proof: Let $a_n = \begin{cases} e^{2\pi i f(n)}, & 1 \leq n \leq N, \\ 0, & \text{otherwise} \end{cases}$

Then $$HS_N = H\sum_{n=1}^{N} a_n$$
$$= \sum_{k=1}^{H}\sum_{n=-H+1}^{N-1} a_{n+k}$$
$$= \sum_{n=-H+1}^{N-1}\sum_{k=1}^{H} a_{n+k}$$

Then

$$|HS_N|^2 \le (N+H-1)\sum_{n=-H+1}^{N-1}\left|\sum_{k=1}^{H} a_{n+k}\right|^2$$ Cauchy-Schwartz

$$= (N+H-1)\sum_{n=-H+1}^{N-1}\sum_{i=1}^{H}\sum_{j=1}^{H} a_{n+i}\overline{a_{n+j}}$$

$$= (N+H-1)\sum_{n=-\infty}^{\infty}\sum_{i=1}^{H}\sum_{j=1}^{H} a_{n+i}\overline{a_{n+j}}$$

$$= (N+H-1)\sum_{i=1}^{H}\sum_{j=1}^{H}\sum_{n=-\infty}^{\infty} a_{n+i}\overline{a_{n+j}}$$

$$= (N+H-1)\sum_{i=1}^{H}\sum_{j=1}^{H}\sum_{n=-\infty}^{\infty} a_{n+i-j}\overline{a_n}$$

$$= (N+H-1)\Big(H\sum_{n=-\infty}^{\infty}|a_n|^2 + \sum_{h=1}^{H-1}(H-h)\sum_{n=-\infty}^{\infty} a_{n+h}\overline{a_n}$$
$$+ \sum_{h=1}^{H-1}(H-h)\sum_{n=-\infty}^{\infty} a_{n-h}\overline{a_n}\Big)$$

$$= (N+H-1)\Big(H\sum_{n=-\infty}^{\infty}|a_n|^2 + \sum_{h=1}^{H-1}(H-h)\sum_{n=-\infty}^{\infty} a_{n+h}\overline{a_n}$$
$$+ \sum_{h=1}^{H-1}(H-h)\sum_{n=-\infty}^{\infty} a_n\overline{a_{n+h}}\Big)$$

$$= (N+H-1)\left(H\sum_{n=-\infty}^{\infty}|a_n|^2 + \sum_{h=1}^{H-1}(H-h)\, 2\sum_{n=-\infty}^{\infty}\mathrm{Re}(a_{n+h}\overline{a_n})\right)$$

$$\leq 2H(N+H-1)\sum_{h=0}^{H-1}\left|\sum_{n=-\infty}^{\infty}a_{n+h}\overline{a_n}\right|$$

$$\leq 4HN\sum_{h=0}^{H-1}\left|\sum_{n=1}^{N-h}a_{n+h}\overline{a_n}\right| \qquad H\leq N$$

$$= 4HN\sum_{h=0}^{H-1}\left|\sum_{n=1}^{N-h}e^{2\pi i\, f(n+h)-f(n)}\right|$$

$\square$

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II. $\langle n^2\alpha\rangle$ is equidistributed if $\alpha\notin\mathbb{Q}$

Proof: By Weyl's Criterion, it suffices to show for any $k\in\mathbb{Z}\setminus\{0\}$,

$$\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i k n^2\alpha} \longrightarrow 0 \quad \text{as} \quad N\to\infty$$

By I, for any $H\leq N$

$$\left|\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i k n^2\alpha}\right|^2 \leq \frac{4}{NH}\sum_{h=0}^{H-1}\left|\sum_{n=1}^{N-h}e^{2\pi i k\alpha[(n+h)^2-n^2]}\right|$$

$$= \frac{4}{NH}\sum_{h=0}^{H-1}\left|\sum_{n=1}^{N-h}e^{2\pi i k h n\alpha}\right|$$

$$= \frac{4}{H} + \frac{4}{H}\sum_{h=1}^{H-1}\left|\sum_{n=1}^{N-h} e^{2\pi i k h n \alpha}\right|$$

$$\leq \frac{4}{H} + \frac{4}{H}\sum_{h=1}^{H-1}\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k h n \alpha}\right|$$

Fix $H$ large s.t. $\frac{4}{H} < \varepsilon$

$$\leq \varepsilon + \varepsilon\sum_{h=1}^{H-1}\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k h n \alpha}\right|$$

For $N$ large enough $\sum_{h=1}^{H-1}\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k h n \alpha}\right| < \varepsilon$

$$\leq \varepsilon + \varepsilon^2$$

Hence, $\frac{1}{N}\sum_{n=1}^{N-1} e^{2\pi i k n^2 \alpha} \to 0$ as $N \to \infty$.

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Remark:

- With a similar argument, one can show if $\langle x_{n+h} - x_n \rangle$ is equidistributed for any $h \in \mathbb{N}$, then $\langle x_n \rangle$ is equidistributed.

- We can use this 'reducing degree' method to show $\langle n^k \alpha \rangle$ is equidistributed for $k \in \mathbb{N}$, $\alpha \notin \mathbb{Q}$.

In fact, $\langle P(n) \rangle$ is equidistributed if one of the coefficients of the polynomial $P(x)$ is irrational.